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Recurrence conditions in space–time

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Abstract. A general discussion of the recurrence properties of the Riemann, Ricci and Weyl tensors is given. Space-times possessing these properties are classified according to the Petrov type of the Weyl tensor and the Segré type of the Ricci tensor. The proofs of some known theorems are shortened and some new results are given.

1. Introduction

In a number of papers, Thompson (1968, 1969a, b, 1970) has discussed certain types of recurrence properties of tensors on Riemannian spaces and has considered the special case of a Lorentzian space–time manifold. More recently, McLenaghan and Leroy (1972) have discussed complex recurrent and conformally symmetric space–times. In this paper some results of the above authors are reconsidered and extended and some new approaches suggested.

Throughout the paper, M will denote a Lorentzian space–time manifold and if $p \in M$, $T_p(M)$ will denote the tangent space to M at p. The symbols ∇ and $\otimes$ represent the covariant derivative operator and tensor product symbols respectively. When local coordinates are used, the notation will be the conventional one.

It will often be useful to introduce a chart containing a null tetrad of vector fields with components l^a, m^a, e^a, f^a where l^a and m^a are non-collinear null vector fields and e^a and f^a are spacelike vector fields, and where the equivalent conditions $l^a m_a = e^a e_a = f^a f_a = 1$ (all other inner products zero) and $g_{ab} = 2l_{(a}m_{b)} + e_a e_b + f_a f_b$ (the completeness relation) hold. Similarly, it will be useful to introduce an orthonormal tetrad of vector fields with components x^a, y^a, z^a, t^a satisfying the equivalent conditions $x^a x_a = y^a y_a = z^a z_a = -t^a t_a = 1$ (all other inner products zero) and $g_{ab} = x_a x_b + y_a y_b + z_a z_b - t_a t_b$.

The algebraic classification of the Weyl and Ricci tensors will be used extensively. Details of the relevant aspects of the Petrov classification can be found in the papers of Sachs (1961) and Bel (1962). Concerning the Ricci tensor, a brief review of its algebraic properties will be presented here. (For more details see Plebanski (1964), Collinson and Shaw (1972) and Hall (1976a).) The Ricci tensor at $p \in M$, considered as a linear map: $T_p(M) \to T_p(M)$ may, on account of the Lorentz signature of M, take one of the four Segré types $\{1, 1, 1, 1\}$, $\{2, 1, 1\}$, $\{3, 1\}$ and $\{z, \bar{z}, 1, 1\}$ (or their degeneracies). In the notation of the last paragraph one can, for the first type, introduce an orthonormal tetrad of members of $T_p(M)$ such that the components of the Ricci tensor at p become

$$R_{ab} = \rho_1 x_a x_b + \rho_2 y_a y_b + \rho_3 z_a z_b - \rho_4 t_a t_b \tag{1.1}$$

where the ρ are the (real) Ricci eigenvalues at p. For the other cases one can, for each type, introduce a null tetrad at p such that in components, one has respectively

$$R_{ab}=2\sigma_1 l_{(a}m_{b)}+\lambda l_a l_b+\sigma_2 e_a e_b+\sigma_3 f_a f_b \qquad (\lambda\neq 0) \tag{1.2}$$

$$R_{ab}=2\alpha_1 l_{(a}m_{b)}+2\mu l_{(a}e_{b)}+\alpha_1 e_a e_b+\alpha_2 f_a f_b \qquad (\mu\neq 0) \tag{1.3}$$

$$R_{ab}=2\beta_1 l_{(a}m_{b)}+\beta_2(l_a l_b-m_a m_b)+\beta_3 e_a e_b+\beta_4 f_a f_b \qquad (\beta_2\neq 0) \tag{1.4}$$

where the σ, α, β, λ and μ are real numbers with λ, μ and β_2 non-zero. One may always choose the null tetrad such that $\mu=1$ in (1.3) whilst in (1.2), if $\lambda>0$ ($\lambda<0$) one may similarly choose $\lambda=1$ ($\lambda=-1$).

It is noted that if a Ricci tensor has two distinct null eigendirections at p, then their associated eigenvalues are equal and the Ricci tensor has Segré type (some degeneracy of) {(1, 1)1, 1}. Also, the Ricci tensor has a unique null eigendirection if and only if it is similar to a Jordan matrix with a non-simple elementary divisor (that is if and only if it is of type {2, 1, 1} or {3, 1} or some degeneracy thereof). Only the types {1, 1, 1, 1} and {2, 1, 1} contain members consistent with the 'energy condition' in general relativity.

Throughout the paper, the Weyl tensor will be assumed well behaved in the sense that in all space–time regions under consideration, the Weyl tensor will be assumed to determine four (not necessarily distinct) null vector fields on the region which, at each point of the region, constitute the full set of Debever–Penrose vectors at that point. Similarly, any bivector will be assumed to determine, throughout its domain of definition, two (not necessarily distinct) null vector fields which, at each point of the domain, constitute the full set of principal null directions of the bivector at the point. Finally, a similar assumption is made for the eigenvectors of the Ricci tensor. For example, if a Ricci tensor is Segré type {2, 1, 1} throughout a region, then it will be assumed that there exists one null and two spacelike vector fields throughout the region which are distinct Ricci eigenvectors at each point of the region. Similar assumptions are made for the other Segré types.

The following is a summary of the main points of the paper. In § 2, some preliminary results are given. In particular, it is shown that a space–time possessing a null (non-null) bivector with vanishing skew derivative is either Petrov type N or O (Petrov type D or O). The Ricci and Riemann tensors for such space–times are determined. In § 3, complex recurrent and conformally symmetric space–times are discussed. It is shown that such (connected) space–times are of the same Petrov type everywhere and a simple proof is given of the theorem of McLenaghan and Leroy which states that such space–times are of Petrov type N or D. Those complex recurrent or conformally symmetric space–times with a gradient recurrence 1-form are shown equivalent to space–times with non-zero Weyl tensors which admit constant bivectors. This leads to a simple generalization of p–p wave space–times. In § 4, Ricci recurrent space–times are discussed and it is shown that only Ricci tensors whose Segré type is (a subtype of) {1, 1, 1, 1} or {2, 1, 1} can be recurrent and that this subtype is the same everywhere if the space–time is connected. Some extensions to conformally flat space–times are given in § 5 whilst some physical applications to plane electromagnetic radiation are discussed in § 6.

2. Preliminary results

Two bivectors (2-forms) with components A_{ab} and B_{ab} in some chart about a point $p\in M$ are said to be in the same dual (equivalence) class of bivectors at p if at p,

$\overset{+}{A}_{ab} = e^{i\theta}\overset{+}{B}_{ab}$ $(0 \leqslant \theta < 2\pi)$, where for any bivector with components T_{ab} in some chart, $\overset{+}{T}_{ab}$ denotes the (complex) self-dual part of T_{ab}, $\overset{+}{T}_{ab} = T_{ab} + i\overset{*}{T}_{ab}$. A property which if possessed by one bivector at p is possessed by all its dual equivalent bivectors at p will be called a dual invariant. One particular dual invariant property of a bivector is that of having vanishing skew derivative, $T_{ab;[cd]} = 0$, as is easily shown by using the Ricci identity.

Suppose that a null bivector N_{ab} has vanishing skew derivative at $p \in M$. At p one has $N_{ab} = 2l_{[a}p_{b]}$ where l^a and p^a are the components in some chart about p of two members of $T_p(M)$ with $l^a l_a = l^a p_a = 0$. The equation $N_{ab;[cd]} = 0$ when expressed in terms of the Ricci identity and contracted first with l^a and then with p^a easily yields $l^a R_{abcd} = 0$, $l^a R_{ab} = 0$ and $p^e R_{becd} = l_b A_{cd}$ for some real bivector A_{ab} which necessarily satisfies $l^a A_{ab} = 0$ and $l_{[a}A_{bc]} = 0$. Thus A_{ab} is proportional to a null bivector and $p^a R_{ab} = 0$. Next, at p, $\overset{*}{N}_{ab} = 2l_{[a}q_{b]}$ where q^a are the components of some member of $T_p(M)$ satisfying $l^a q_a = p^a q_a = 0$. Since at p, $N_{ab;[cd]} = 0 \Rightarrow \overset{*}{N}_{ab;[cd]} = 0$, similar arguments to those given above imply $R_{ab}q^b = 0$ and so the Ricci tensor is either zero or else has Segré type {(2, 1, 1)} with vanishing eigenvalue at p and canonical form

$$R_{ab} = Al_a l_b \tag{2.1}$$

where $A \in \mathbb{R}$. It now follows from equation (A.1) that $l^a \overset{+}{C}_{abcd} = 0$ at p where $\overset{+}{C}_{abcd} = C_{abcd} + i\overset{*}{C}_{abcd}$ are the components of the complex self-dual Weyl tensor and so *the Weyl tensor is either zero or Petrov type N at p.* The form of the Riemann tensor at p can now be evaluated using (2.1), the Petrov type N canonical form (Sachs 1961) and the null tetrad completeness relation for the components of the metric tensor. One finds

$$R_{abcd} = A_1 P_{ab}P_{cd} + A_2 \overset{*}{P}_{ab}\overset{*}{P}_{cd} \tag{2.2}$$

for real numbers A_1 and A_2 and a null bivector P_{ab} with principal null direction proportional to l^a.

Conversely, if at p the Weyl tensor is zero or Petrov type N with repeated principal null direction along l^a and the Ricci tensor satisfies (2.1), then any null bivector N_{ab} with principal null direction along l^a has vanishing skew derivative at p. In fact, (2.1) implies

$$N_{ab;[cd]} = N_{e[a}R_{b]}{}^{e}{}_{cd} = N_{e[a}C_{b]}{}^{e}{}_{cd} \tag{2.3}$$

and the final term is easily shown to be zero if the Weyl tensor is zero or Petrov type N.

Suppose now that a non-null bivector P_{ab} has vanishing skew derivative at $p \in M$ and let l^a and m^a be the components, in some chart about p, of the principal null directions of P_{ab} at p. Since the property $P_{ab;[cd]} = 0$ is dual invariant and since any dual class of bivectors contains a simple bivector, one need only consider simple bivectors with vanishing skew derivative. It is easy to show the existence of a real number T such that the bivector $2Tl_{[a}m_{b]}$ is in the same dual class as P_{ab} at p. Then the Ricci identity applied to this bivector at p, upon contraction first with l^a and then with m^a, yields $l^a R_{abcd} = l_b A_{cd}$ and $m^a R_{abcd} = -m_b A_{cd}$ for some bivector A_{ab}. Hence $l_{[b}A_{cd]} = m_{[b}A_{cd]} = 0$ and so $A_{ab} = 2\alpha m_{[a}l_{b]}$ for some $\alpha \in \mathbb{R}$. So l^a and m^a are null eigenvectors of the Ricci tensor with (necessarily) equal eigenvalues from which it follows that the Ricci tensor has only simple elementary divisors. One then finds

$$l^a l^c C_{abcd} = \tfrac{1}{6}Rl_b l_d \qquad m^a m^c C_{abcd} = \tfrac{1}{6}Rm_b m_d. \tag{2.4}$$

Next, if one calculates the quantity $l^a C_{abcd}\eta^{cdmn} l_m$ where η_{abcd} is the alternating symbol, the above results yield

$$\overset{*}{C}_{abcd} l^a l^c = 0 \qquad \overset{*}{C}_{abcd} m^a m^c = 0. \tag{2.5}$$

Thus the Weyl tensor is either zero or Petrov type D at p, the former case being characterized by the condition $R = 0$ at p. A straightforward calculation using the above results and the Petrov type D canonical form for the Weyl tensor shows that the two-dimensional subspace of $T_p(M)$ consisting of (spacelike) vectors orthogonal to l^a and m^a is an eigen-subspace of the Ricci tensor with eigenvalue $\frac{1}{2}R - \alpha$. So the Ricci tensor has Segré type {(1, 1)(1, 1)} (or some degeneracy of this type) and canonical form at p

$$R_{ab} = 2\alpha l_{(a} m_{b)} + (\tfrac{1}{2}R - \alpha)(e_a e_b + f_a f_b) \tag{2.6}$$

in the null tetrad notation of § 1. The Riemann tensor can then be shown to take the form (see appendix 1):

$$R_{abcd} = (\alpha - \tfrac{1}{2}R)\overset{*}{S}_{ab}\overset{*}{S}_{cd} + \alpha S_{ab} S_{cd} \tag{2.7}$$

where $S_{ab} = 2l_{[a} m_{b]}$.

3. Complex recurrent space–times

If U is some open subset of the space–time manifold M, then the Weyl tensor is called *complex recurrent* on U if at each point of U the complex self-dual Weyl tensor $\overset{+}{C}$ is nowhere zero and satisfies

$$\nabla \overset{+}{C} = \overset{+}{C} \otimes k \tag{3.1}$$

where k is a nowhere zero complex 1-form. Within a chart in U, equation (3.1) takes the component form $\overset{+}{C}_{abcd;e} = \overset{+}{C}_{abcd} k_e$. If (3.1) holds with k the zero 1-form on U, but with C nowhere zero on U, the Weyl tensor is called *conformally symmetric* on U. McLenaghan and Leroy (1972) have shown that if the Weyl tensor is complex recurrent or conformally symmetric on U, then it is either Petrov type N or D throughout U. Their proof however, was based on the initial assumption that for such spaces the Petrov type of the Weyl tensor was the same at each point of U and consisted of checking the consistency of each Petrov type with (3.1)†. A proof is now given which avoids this assumption and gives the result more quickly when U is connected in the manifold topology. Suppose then that U is a connected open submanifold of M and let p and q be any two points of U. Then it follows that there exists a piecewise smooth path c from p to q. The Lorentz connection on M allows one to set up an isomorphism between the tensor fibres over p and those over q according to the usual parallel propagation. Since the path c may be broken up into a finite number of smooth pieces each of which lies inside a chart of U, it is easily seen that in the proof outlined below one may suppose without loss of generality that the path c is smooth and lies entirely within a chart of U. Now if the Weyl tensor is complex recurrent or conformally symmetric on U, then (3.1)

†A proof similar to the one given by McLenaghan and Leroy and which also makes such an assumption can be achieved by writing the complex Weyl tensor in the canonical forms given by Sachs (1961) and checking the consistency of each Petrov type with (3.1) using the canonical bivector differential relations given by Robinson and Schild (1963).

shows that the Weyl tensor at q is complex proportional to the parallel propagate at q of the Weyl tensor at p. The constancy of the Petrov type of the Weyl tensor over U is now straightforward to establish. Suppose for example that the Weyl tensor is Petrov type N at p. By Bel's criteria (Bel 1962), this is equivalent to the existence of a (necessarily null) non-zero vector $l \in T_p(M)$ such that in components, $l^a \overset{+}{C}_{abcd} = 0$. The parallel propagate of l along c to q would then furnish a non-zero null vector $l' \in T_q(M)$ satisfying $l'^a \overset{+}{C'}_{abcd} = 0$ where C' denotes the parallel propagate of the Weyl tensor along c. The above remarks combine to yield $l'^a \overset{+}{C}_{abcd} = 0$ at q and Bel's criteria are again used to conclude that the Weyl tensor is Petrov type N at q. Similar remarks apply to the other Petrov types using the appropriate Bel criteria for the type. In the type N case, it follows from the assumption concerning the existence of Debever–Penrose vector fields that there exists a null vector field on U which everywhere agrees with the principal null direction of the Weyl tensor. In the type III case, two Debever–Penrose vector fields will exist on U and a continuity argument using the above parallel propagation ideas shows that one of these vector fields will everywhere agree with the repeated Debever–Penrose vector of the Weyl tensor and the other with the non-repeated Debever–Penrose vector. Similar remarks apply to the other Petrov types.

To see which Petrov types actually occur, suppose C is complex recurrent or conformally symmetric on U. Let l be a Debever–Penrose vector field on U and let $p \in U$. If c is a smooth curve from p lying within a chart of U then by arguments similar to those given above, one concludes that at any point q on c, the parallel propagate of l_p at q, l'_q, is a Debever–Penrose vector at q. Next, since there are a finite number of independent Debever–Penrose vectors at each point of U, one can arrange, by choosing q sufficiently close to p and using a continuity argument, that l_q and l'_q are parallel. Since the initial direction of c at p was arbitrary, one concludes that l is *recurrent* at p, $\nabla l = l \otimes w$ for some 1-form w. In components, this recurrence condition becomes $l_{a;b} = l_a w_b$ and so the Ricci identity yields $l^a R_{abcd} = l_b A_{cd}$ for some bivector A_{ab} which must then be a simple bivector whose blade contains l^a. It then follows that l^a is a Ricci eigenvector and that $l^a l^c R_{abc[d} l_{e]} = 0$. This information leads to $l^a l^c \overset{+}{C}_{abc[d} l_{e]} = 0$ whence l is a repeated Debever–Penrose vector. This shows that if the Weyl tensor is complex recurrent or conformally symmetric, then it has only repeated Debever–Penrose vectors and so must be of Petrov type N or D†. Further, these (repeated) Debever–Penrose vectors must be recurrent.

In the type N case, if l is the repeated principal null direction of the Weyl tensor, the local equation $l^a R_{abcd} = l_b A_{cd}$ shows that at each point, either: (i) $A_{ab} = 0$, (ii) A_{ab} is a null bivector satisfying $l^a A_{ab} = 0$ or (iii) A_{ab} is a non-null bivector whose (timelike) blade contains the direction l. If $\mathscr{C}$ is a smoothly contractible chart domain of U such that (i) holds throughout $\mathscr{C}$ then it is readily shown that, in $\mathscr{C}$, $R = 0$ and l is a Ricci eigendirection with zero eigenvalue. Also, the conditions $l_{a;b} = l_a w_b$ and $l^a R_{abcd} = 0$ together with the Ricci identity and the Poincaré lemma show that w_a is a gradient. $w_a = w_{,a}$ and so l^a is proportional to the constant null vector field $e^{-w} l^a$. The Ricci tensor takes the form (2.1) and the Riemann tensor the form (2.2). A constant null bivector with principal null direction parallel to l is admitted (see appendix 2). If condition (ii) holds throughout $\mathscr{C}$ then again $R = 0$ in $\mathscr{C}$ and l is again a Ricci eigendirection with zero eigenvalue. The Ricci tensor takes the Segré type $\{(3, 1)\}$ with zero eigenvalue. If condition (iii) holds throughout $\mathscr{C}$, again $R = 0$ in $\mathscr{C}$ and l is a Ricci

† A similar proof which considers the eigenbivectors of the Weyl tensor is also possible.

eigendirection with non-zero eigenvalue. The Ricci tensor takes either Segré type $\{2,(1,1)\}$ or $\{(1,1)(1,1)\}$ where in both cases, the two non-zero eigenvalues differ only in sign (see appendix 2).

In the type D case, if l^a and m^a represent, in local coordinates, the two (recurrent) principal null directions of the Weyl tensor, then on scaling so that $l^a m_a = 1$, one sees that the non-null bivector $2l_{[a}m_{b]}$ is constant. It then follows from the results of § 2 that the Ricci and Riemann tensors satisfy (2.6) and (2.7) where in (2.7), S_{ab} is constant. The Weyl tensor invariant is $\frac{1}{6}R$ and the recurrence vector in (3.1) satisfies, in local coordinates, $k_a = (\ln|R|)_{,a}$. So the conformally symmetric case is characterized by $R = \text{constant} \neq 0$. Clearly there are no type D complex recurrent or conformally symmetric vacuum space–times (cf Ehlers and Kundt 1962).

Now let $\mathscr{C}$ be a connected chart domain. Then the following two conditions are equivalent.

(*a*) The Weyl tensor is conformally symmetric or complex recurrent with a gradient recurrence 1-form on $\mathscr{C}$.

(*b*) The Weyl tensor is nowhere zero on $\mathscr{C}$ and $\mathscr{C}$ admits a constant bivector.

To see this, note that if (*a*) holds and the Weyl tensor is type N on $\mathscr{C}$ then any complex null self-dual bivector V_{ab} whose principal null direction coincides with the (recurrent) principal null direction of the Weyl tensor satisfies $V_{ab;c} = V_{ab}\alpha_c$ for some 1-form α_a. The complex Weyl tensor can then be written as $\overset{+}{C}_{abcd} = C' V_{ab} V_{cd}$ where C' is complex and the recurrence condition (3.1) gives

$$k_a = (C')^{-1} C'_{,a} + 2\alpha_a. \tag{3.2}$$

So k_a is a gradient $\Leftrightarrow \alpha_a$ is a gradient, $\alpha_a = \alpha_{,a}$. But this latter condition means that V_{ab} is (complex) proportional to the constant complex null bivector $e^{-\alpha} V_{ab}$. Conversely, if (*b*) holds then it is firstly remarked that parallel propagation arguments similar to those presented earlier show that if a bivector is constant on $\mathscr{C}$ then it is either null throughout $\mathscr{C}$ or non-null throughout $\mathscr{C}$. Suppose then that a constant null bivector V_{ab} is admitted. Then by § 2, the Weyl tensor is Petrov type N on $\mathscr{C}$ and takes the above canonical form. It then follows that (3.1) holds with k_a a gradient on $\mathscr{C}$. In the type D case when (3.1) holds, then necessarily k_a is a gradient and as follows from above a constant non-null bivector is always admitted. Conversely if (*b*) holds and a constant non-null bivector is admitted on $\mathscr{C}$ then, from § 2, the Weyl tensor is Petrov type D and the principal null directions of the Weyl tensor coincide with those of the bivector and are recurrent. It then follows from the canonical Petrov type D form for the Weyl tensor (Sachs 1961) that (3.1) holds with k_a a gradient.

In the type N case, several characterizations can be given. In fact the following are equivalent if $\mathscr{C}$ is connected and smoothly contractible.

(*a'*) The Weyl tensor is nowhere zero on $\mathscr{C}$ and $\mathscr{C}$ admits a constant null bivector.

(*b'*) The Weyl tensor is conformally symmetric or complex recurrent on $\mathscr{C}$ with a gradient recurrence 1-form and is Petrov type N.

(*c'*) The Weyl tensor is conformally symmetric or complex recurrent on $\mathscr{C}$ and on $\mathscr{C}$ the Ricci tensor takes the form (2.1) for some null vector field l^a on $\mathscr{C}$.

(*d'*) The Weyl tensor is Petrov type N on $\mathscr{C}$ and $\mathscr{C}$ admits a constant null vector field.

That (*a'*) and (*b'*) are equivalent has already been established. To show (*a'*)$\Rightarrow$(*d'*) it is sufficient to recall the results of § 2 and to note that the principal null direction of a constant null bivector contains a constant null vector. To show (*d'*)$\Rightarrow$(*c'*) note that if l^a are the components of the constant null vector field then the Ricci identity gives

$l^a R_{abcd} = 0$ and $l^a R_{ab} = 0$. It easily follows that l^a must be the (unique) principal null direction of the Weyl tensor, $l^a \overset{+}{C}_{abcd} = 0$ and from (A.1) one finds that (2.1) holds. Finally to show that $(c') \Rightarrow (a')$ note that if (2.1) holds in $\mathscr{C}$ for some null vector field l^a, then from the results of this section it follows that the Weyl tensor is Petrov type N and l^a is recurrent. So any complex self-dual null bivector V_{ab} with l^a as principal null direction satisfies $V_{ab;c} = V_{ab}\alpha_c$ for some 1-form α_a and has vanishing skew derivative by (2.3). Then since $\mathscr{C}$ is smoothly contractible, α_a is a gradient and a constant null bivector is admitted.

It should be remarked that any of the conditions $(a'), (b'), (c'), (d')$ above is equivalent to the type N case (i) condition mentioned earlier holding everywhere in $\mathscr{C}$. Also, the similarities between the special case characterized by the above four equivalent conditions and the general vacuum complex recurrent case are now apparent (Hall 1974)†.

Finally, it is pointed out that in the complex recurrent and conformally symmetric case, a detailed study of the exact form of the metric tensor has been undertaken (McLenaghan and Leroy 1972, see also McLenaghan and Thompson 1972a, b, Leroy and McLenaghan 1973 and Collinson and Söler 1973).

4. Ricci recurrent space–times

If U is some open subset of the space–time manifold M then the Ricci tensor R is called *recurrent* on U if R is non-zero on U and $\nabla R = R \otimes \theta$‡ for some 1-form θ on U. In local coordinates this gives $R_{ab;c} = R_{ab}\theta_c$ where θ_a are the components of θ. In order to show which of the algebraic types of Ricci tensor discussed in § 1 are consistent with the Ricci tensor being recurrent, it is first shown that if U is connected and R recurrent on U, then the Segré type of R is the same throughout U. The proof is similar to that given for the Weyl tensor in the last section. One shows by the parallel propagation of eigenvectors that if $p, q \in U$, then to each spacelike (respectively timelike, null) eigendirection of R at p there corresponds a spacelike (respectively timelike, null) eigendirection of R at q and conversely. The result then follows. A check on which of the algebraic Ricci types are consistent with the recurrence condition can be obtained directly, after a lengthy argument, from the canonical Ricci types of § 1. It turns out (Hall 1976b) that no Ricci tensor of Segré type $\{3, 1\}$ or $\{z, \bar{z}, 1, 1\}$ can be recurrent on U, that if a Ricci tensor has Segré type $\{2, 1, 1\}$ and is recurrent on U then all its eigenvalues are zero (and so takes the local form (2.1)) and that if the Ricci tensor has Segré type $\{1, 1, 1, 1\}$ and is recurrent on U then, locally, (1.1) holds where throughout the relevant chart, the ρ satisfy one of the following conditions:

$$\begin{array}{llll} (a) & 0 \neq \rho_a = \rho_b = \rho_c = \rho_d & (b) & \rho_d = 0 \neq \rho_a = \rho_b = \rho_c \\ (c) & 0 \neq \rho_a = \rho_b \neq \rho_c = \rho_d \neq 0 & (d) & \rho_c = \rho_d = 0 \neq \rho_a = \rho_b \end{array} \tag{4.1}$$

where in (4.1), (a, b, c, d) is some permutation of $(1, 2, 3, 4)$.

A much quicker proof of this fact can be obtained by using the methods of § 3. If the Ricci tensor is type $\{z, \bar{z}, 1, 1\}$ on U then the eigendirections of the Ricci tensor at any point of U are either parallel to e^a or to f^a or, if $\beta_3 = \beta_4$, to some (any) linear combination of e^a and f^a, in the notation of (1.4). The parallel propagation argument

† A brief discussion is given in appendix 3.

‡ Here, θ is not restricted to being nowhere zero on U.

(remembering the relation $e^a e_a = f^a f_a = 1$) then yields the local equations $e_{e;b} = f_a p_b$ and $f_{a;b} = -e_a p_b$ for some vector field p^a. The bivector $2e_{[a}f_{b]}$ is then a non-null constant bivector and § 2 shows that this is a contradiction. If the Ricci tensor is type $\{3, 1\}$ on U then, in the notation of (1.3), l is the unique null eigendirection of R and is hence recurrent. Similarly the eigendirections of R at any point are either parallel to l^a or f^a or, if $\alpha_1 = \alpha_2$, to some (any) linear combination of them. Again, remembering the orthogonality relations on the null tetrad in (1.3), one obtains $l_{a;b} = l_a q_b$, $e_{a;b} = l_a U_b$, $f_{a;b} = l_a V_b$. On applying the Ricci identities to e_a and f_a one finds $e^a R_{abcd} = l_b X_{cd}$ and $f^a R_{abcd} = l_b Y_{cd}$ for simple bivectors X_{ab} and Y_{ab} whose blades contain the direction l. A contraction then shows that both $e^a R_{ab}$ and $f^a R_{ab}$ are parallel to l_b and this implies from (1.3) that $\alpha_1 = \alpha_2 = 0$. A simple calculation then shows that no Ricci tensor of the form (1.3) with $\alpha_1 = \alpha_2 = 0$ can be recurrent. In fact a simple substitution into the recurrence equation reveals that $e_{a;b} = 0$ and so from the Ricci identity $e^a R_{ab} = 0$, which implies that $\mu = 0$. In the $\{2, 1, 1\}$ case again l is the unique null eigendirection and is hence recurrent. Suppose now that σ_1, σ_2 and σ_3 in (1.2) are distinct at some point p. Then they are distinct in some neighbourhood of p and so the parallel propagates of $(e)_p$ and $(f)_p$ at some point q close to p must be parallel to $(e)_q$ and $(f)_q$ respectively. So e^a and f^a, on account of the relations $e^a e_a = f^a f_a = 1$, must be constant at p whence the Ricci identity yields the contradiction $\sigma_2 = \sigma_3 = 0$. If at p, $\sigma_1 \neq \sigma_2 = \sigma_3$ then an argument similar to one given in the $\{z, \bar{z}, 1, 1\}$ case shows the existence of a constant non-null bivector and hence a contradiction. Similar arguments remove the cases $\sigma_1 = \sigma_2 \neq \sigma_3$ and $\sigma_1 = \sigma_3 \neq \sigma_2$. Hence the only case left is $\sigma_1 = \sigma_2 = \sigma_3$ ($= \sigma$ say). In this case one has the local equation $R_{ab} = \lambda l_a l_b + \sigma g_{ab}$. If $\sigma \neq 0$, a direct substitution into the recurrence condition shows that l^a is parallel to a constant null vector which, from the Ricci identity, implies the contradiction $\sigma = 0$. For the $\{1, 1, 1, 1\}$ case, if one of the eigenvalues ρ_1, ρ_2, ρ_3, ρ_4, say ρ_1, was distinct from the rest at a point p then the above arguments show that the corresponding eigenvector in the tetrad is constant and so $\rho_1 = 0$ at p. The remainder of the eigenvalues must be equal (and non-zero) otherwise one of these eigenvalues would be zero at p, contradicting the initial statement about ρ_1. The only other possibilities are when all the eigenvalues are equal or when they are equal in pairs. Equation (4.1) then results.

In the $\{2, 1, 1\}$ case, a recurrent Ricci tensor takes the local form (2.1) with l^a recurrent and so by methods similar to those in § 3 one finds that the Weyl tensor is of Petrov type O, N or III, where in the latter two cases, l is the repeated principal null direction of the Weyl tensor. In the type N case, the Weyl tensor is complex recurrent or conformally symmetric since, for type N spaces, the equation (3.1) is equivalent to the repeated principal null direction of the Weyl tensor being recurrent. The Riemann tensor is not necessarily recurrent or constant but if it is either, then it readily follows that the Weyl tensor satisfies (3.1) and so from the results of § 3, it is necessarily Petrov type N or O. The general form of the Riemann tensor is readily evaluated in all cases.

In the type $\{1, 1, 1, 1\}$ case, equation (4.1*a*) represents an Einstein space and the ρ and the Ricci tensor are constant. In the case when (4.1*b*) holds the local coordinate form of the Riemann tensor can be easily constructed. Suppose, for example, we have $\rho_4 = 0$. Then since ρ_4 is distinct from the other eigenvectors, it follows from remarks above that the corresponding eigenvector field t^a is constant and so the Ricci identity gives $t^a R_{abcd} = 0$. Next, one notes that the Riemann tensor can be represented as a linear combination of outer products of pairs of the six canonical simple bivectors generated by the eigenvectors x^a, y^a, z^a, t^a. The above condition on t^a means that no bivector whose blade contains the direction t^a will occur in such an expression. These facts together with the remark that the Riemann tensor so constructed must reduce to the Ricci tensor given by (4.1b) lead to the following result for the case $\rho_4 = 0$:

$$R_{abcd} = -2\rho(x_{[a}y_{b]}x_{[c}y_{d]} + x_{[a}z_{b]}x_{[c}z_{d]} + y_{[a}z_{b]}y_{[c}z_{d]}) \tag{4.2}$$

where $\rho = \rho_1 = \rho_2 = \rho_3$. The other cases are similar. The derivatives of the canonical bivectors in (4.2) can be found from the tetrad derivatives

$$\begin{aligned} x_{a;b} &= y_a\beta_b + z_a\gamma_b & y_{a;b} &= -x_a\beta_b + z_a\delta_b \\ z_{a;b} &= -x_a\gamma_b - y_a\delta_b & t_{a;b} &= 0 \end{aligned} \tag{4.3}$$

for vector fields with local components β^a, γ^a and δ^a, which are readily obtainable from the orthonormality relations on the tetrad and the condition $t_{a;b} = 0$. It can then be checked that the Bianchi identity $R_{ab[cd;e]} = 0$ implies that ρ and that the Riemann, Ricci and hence Weyl tensors are all constant. A comparison with § 3 shows that the Weyl tensor is necessarily zero since otherwise it would be conformally symmetric and its Ricci tensor type would be inconsistent with (4.1b). If (4.1c) holds then arguments similar to those used in rejecting the $\{z, \bar{z}, 1, 1\}$ case above show that a constant non-null bivector is admitted. For example, if $\rho_1 = \rho_2$ and $\rho_3 = \rho_4$ then the bivectors $2x_{[a}y_{b]}$ and $2z_{[a}t_{b]}$ are constant. In this case the results of § 3 show that in local coordinates, the Riemann tensor becomes

$$R_{abcd} = -4\rho_1 x_{[a}y_{b]}x_{[c}y_{d]} + 4\rho_3 z_{[a}t_{b]}z_{[c}t_{d]} \tag{4.4}$$

and that the Weyl tensor is Petrov type D or O according as $\rho_1 + \rho_3 \neq 0$ or $\rho_1 + \rho_3 = 0$. The recurrence condition on the Ricci tensor together with the Bianchi identity used above then show that ρ_1 and ρ_3 are constant and that the Ricci, Riemann and Weyl tensors are all constant. The other cases are similar. Finally if (4.1d) holds then again a constant non-null bivector is admitted and by similar arguments to the above one concludes that the Ricci, Riemann and Weyl tensors are all recurrent and are constant if and only if the non-vanishing ρ is constant. The recurrence 1-form θ is a gradient, proportional to the gradient of the non-vanishing ρ, and the vector field associated with θ with local components θ^i is a Ricci eigenvector (cf Roter 1962). The Weyl tensor is of Petrov type D.

In the case where the Ricci scalar is non-zero, the above results enable the following statement to be made. Firstly, if the Ricci tensor is recurrent on U with θ nowhere zero on U, then the Riemann tensor is recurrent (and nowhere constant) on U. Secondly, if the Ricci tensor is not proportional to the metric tensor on U, then, on U, a constant Ricci tensor implies a constant Riemann tensor.

5. Conformally flat space–times

The Riemann tensor $\mathcal{R}$ is called *recurrent* on U if $\mathcal{R}$ is nowhere zero on U and throughout U,

$$\nabla\mathcal{R} = \mathcal{R} \otimes \omega \tag{5.1}$$

where ω is a nowhere zero 1-form on U. If (5.1) holds with $\omega = 0$ on U, then $\mathcal{R}$ is called *symmetric* on U. In local coordinates, (5.1) reads $R_{abcd;e} = R_{abcd}\omega_e$. It follows from

Walker's theorem (Walker 1950) that since any manifold is locally smoothly contractible, ω is locally a gradient. If $\mathscr{R}$ is recurrent (symmetric) and the Weyl tensor is non-zero on U then the Weyl tensor is complex recurrent with a recurrence 1-form which is real and locally a gradient (conformally symmetric) and the Ricci tensor is recurrent. In §§ 3 and 4 information is supplied concerning the Petrov type and Ricci tensor type of such space–times and when such space–times admit constant bivectors and constant null vectors.

Suppose now that the Weyl tensor is zero on a chart domain $\mathscr{C}$ and that $\mathscr{C}$ admits a constant non-null bivector. Let l and m be the principal null vector fields of the constant bivector on U. If l^a and m^a are the components of these vector fields, scaled so that $l^a m_a = 1$, then by techniques similar to those given earlier one finds that l^a and m^a are recurrent with recurrence vectors differing only in sign. Hence $l_{(a}m_{b);c} = 0$. It then follows from the completeness relation of § 1, the expressions (2.6) and (2.7) for the Ricci and Riemann tensors and the contracted Bianchi identity $R^b_{a;b} = 0$ (since now $(2.4) \Rightarrow R = 0$) that α is a constant in (2.6) and then (2.7) implies that $R_{abcd;e} = 0$. So the Riemann tensor, if non-zero on $\mathscr{C}$, is symmetric on $\mathscr{C}$.

Suppose now that the Weyl tensor is zero on $\mathscr{C}$ and that $\mathscr{C}$ admits a constant null bivector with principal null vector field l on U. It then follows from § 2 that the Ricci tensor takes the form (2.1) where l^a may be chosen to be constant. Hence the Ricci tensor if non-zero on $\mathscr{C}$ is recurrent on $\mathscr{C}$. The vanishing of the Weyl tensor on $\mathscr{C}$ then means that the Riemann tensor (if non-zero on $\mathscr{C}$) is either recurrent or symmetric on $\mathscr{C}$. So if the Weyl tensor is zero on $\mathscr{C}$ and $\mathscr{C}$ admits a constant bevector then the Riemann tensor, if non-zero on $\mathscr{C}$ is either recurrent or symmetric on $\mathscr{C}$.

The following converse is available. Suppose the Weyl tensor is zero on $\mathscr{C}$, that $\mathscr{C}$ is smoothly contractible and that the Riemann tensor is recurrent on $\mathscr{C}$. Then the Ricci tensor is recurrent on $\mathscr{C}$ with a nowhere zero recurrence 1-form ω and it follows from § 4 that the Ricci tensor takes the form (2.1) on $\mathscr{C}$ and l^a is recurrent. The conformally flat Bianchi identity $R_{a[b;c]} = 0$ then implies that l_a is parallel to ω_a. So any complex null bivector V_{ab} on $\mathscr{C}$ with principal null direction along l^a satisfies $V_{ab;c} = V_{ab}\gamma_c$ for some 1-form on $\mathscr{C}$ with components γ_a. Finally V_{ab} has vanishing skew derivative from (2.3) and manipulations similar to those of § 3, using the Poincaré lemma, show that $\mathscr{C}$ admits a constant null bivector. The existence of a Ricci tensor of the type (4.1b) together with the results of § 2 would show that this converse is false if the Riemann tensor is symmetric rather than recurrent on $\mathscr{C}$.

Finally it is noted that if the Weyl tensor is conformally symmetric and Petrov type D on some open subset U of M then the Riemann tensor is symmetric on U (McLenaghan and Leroy 1972). This immediately follows in the present formalism by noting that the conditions of the result imply that in local coordinates $R = \text{constant} \neq 0$. The identity $R^b_{a;b} = 0$ then shows that α is a constant in (2.6) and then (2.7) implies that $R_{abcd;e} = 0$.

6. Applications

For the type N complex recurrent fields which satisfy the equivalent conditions (a'), (b'), (c'), (d') of § 3, one can choose coordinates such that the line element is

$$ds^2 = dx^2 + dy^2 + 2du\,dv + H\,du^2. \tag{6.1}$$

This follows since such fields admit a constant null bivector (Ehlers and Kundt 1962). In (6.1) x and y are coordinates in a spacelike 2-surface orthogonal to l^a, v is an affine parameter along l^a and $l_a = u_{,a}$. The function H depends on x, y and u only. Under certain conditions on H, such fields may be interpreted as source-free null Maxwell fields, where the relevant tensors take the form

$$\overset{+}{F}_{ab} = \psi H_{ab} \qquad \overset{+}{C}_{abcd} = \phi H_{ab} H_{cd} \qquad R_{ab} = A l_a l_b. \tag{6.2}$$

In (6.2), H_{ab} is a constant complex null bivector, l_a a constant real null vector, $\overset{+}{F}_{ab}$ the complex self-dual Maxwell bivector, A a real function and ψ and ϕ complex (amplitude-polarization) functions. Maxwell's source-free equations show that $\psi = \psi(u, z)$ and is an analytic function of $z = x + iy$ for each value of u. The rays of the field, represented by the paths of the vector field l^a, form an expansion-free, twist-free, shear-free, rotation-free null geodesic congruence. A field which intuitively might represent a plane null Maxwell radiation field occurs where ψ and ϕ depend on the phase u only. In fact one need only make the assumption $\phi = \phi(u)$ because then the Bianchi identities and the Einstein–Maxwell equations imply $\psi = \psi(u)$ and also $A = A(u)$. By following closely the arguments of Ehlers and Kundt (1962), one arrives at the conclusion that under this assumption, the quantities $H_{,xx}$ $H_{,yy}$ and $H_{,xy}$ depend only on u, whence (cf McLenaghan and Leroy 1972),

$$H = A_1 x^2 + A_2 y^2 + A_3 xy + A_4 x + A_5 y + A_6 \tag{6.3}$$

with the A_i $(1 \leq i \leq 6)$ depending only on u. A u-dependent translation in the xy plane together with a change of affine origin of the form $v \to v' = v + f(x, y, u)$ may now be made which preserves the general form (6.3) but which makes $A_4 = A_5 = A_6 = 0$. One can then show that such metrics admit (at least) the five-parameter group of motions typical of 'plane waves' whose paths in the five-parameter case are contained in the minimum invariant varieties $u =$ constant. Further, such metrics are complete. The proof of this latter result follows from the fact that given any point p and tangent vector ω at p, one can perform a coordinate transformation which preserves the metric (6.3) with $A_4 = A_5 = A_6 = 0$ and is such that in the new coordinates, p is the origin and the vector ω is tangent either to the subspace $u = v = 0$ or to the subspace $x = y = 0$. The proof is completed by noting that both these latter subspaces are totally geodesic and complete.

The null Maxwell bivector in the above plane wave field was complex recurrent and proportional to a constant null bivector,

$$\text{(i)}\ \overset{+}{F}_{ab;c} = \overset{+}{F}_{ab} f_c \qquad \text{(ii)}\ \overset{+}{F}_{ab} = \psi H_{ab} \tag{6.4}$$

for some 1-form with components f_a. Although (6.4i) implies (6.4ii) if $\overset{+}{F}_{ab}$ is a non-null bivector, this is not true if $\overset{+}{F}_{ab}$ is null. (This result follows since a null bivector is characterized by the condition $\overset{+}{T}_{ab}\overset{+}{T}^{ab} = 0$.) For null Maxwell bivectors, the condition (6.4i) is equivalent to the principal null direction of $\overset{+}{F}_{ab}$ being recurrent which, in turn, is equivalent to this principal null direction being geodesic, with vanishing expansion, twist, shear and rotation. But from (2.1), such fields could be Petrov type III, N or O and from (2.3) only those of type N or O satisfy (6.4ii). The above discussion of plane waves concerned those type N null Maxwell fields satisfying (6.4i) together with the planeness condition embodied in the assumption on the function ϕ. Indeed, these are the only assumptions required since the type N condition and (6.4i) imply that the principal null direction of $\overset{+}{F}_{ab}$ (and $\overset{+}{C}_{abcd}$) is recurrent whence $\overset{+}{C}_{abcd}$ is also (complex) recurrent. Finally, the field equation (2.1) shows that the condition (c') of § 3 holds. Equation (6.4ii) easily follows.

It is perhaps of interest to remark that *for conformally flat null Maxwell fields,* (6.4i *and* ii) *automatically hold* since for such fields the Maxwell principal null direction is recurrent (in fact proportional to a constant null vector) (Stephani 1967, McLenaghan *et al* 1975) and so (6.4i) follows immediately, with (6.4ii) being a consequence of (2.3)†. Further, the planeness requirements are also automatic since in (6.2) we have $\phi = 0$ and the conformally flat Bianchi identity $R_{a[b;c]} = 0$ implies that $A = A(u)$. The Einstein–Maxwell equations and the fact that $\psi(u, z)$ is analytic in z then gives $\psi = \psi(u)$. Equation (6.1) holds where the coordinates can be chosen so that H satisfies (6.3) with $A_3 = A_4 = A_5 = A_6 = 0$ and A_1 $(= A_2)$ a function of u only. This latter metric admits a six-parameter group of motions, five parameters of which are similar to the five discussed earlier. The sixth parameter is the wave surface rotation $z \to z' = z\,\mathrm{e}^{\mathrm{i}\alpha}$ ($z = x + \mathrm{i}y$ and α a constant). These fields are also complete. The extra Killing parameter should be compared with the result (due to Szekeres 1965) that a circular cloud of neutral test particles would be dispersed *symmetrically* by a conformally flat null Maxwell field whereas, for example, a type N null Maxwell field would elliptically *shear* such a cloud.

Appendix 1

From the null tetrad (l^a, m^a, e^a, f^a) used in (2.6) one constructs the complex null tetrad $(l^a, m^a, n^a, \bar{n}^a)$ where $\sqrt{2}n^a = e^a + \mathrm{i}f^a$. Thus one has the equivalent conditions $l^a m_a = n^a \bar{n}_a = 1$ (all other inner products zero) and $g_{ab} = 2l_{(a}m_{b)} + 2\bar{n}_{(a}n_{b)}$. By using the complex self-dual bivectors $V_{ab} = 2l_{[a}\bar{n}_{b]}$, $U_{ab} = 2m_{[a}n_{b]}$ and $M_{ab} = 2l_{[a}m_{b]} + 2\bar{n}_{[a}n_{b]}$ one can construct the bivector completeness relation (Sachs 1961)

$$g_{a[c}g_{d]b} = \mathrm{Re}\,(V_{ab}U_{cd} + U_{ab}V_{cd} - \tfrac{1}{2}M_{ab}M_{cd}).$$

Since in the case considered one has $R_{abcd}V^{cd} = R_{abcd}U^{cd} = 0$, it is now straightforward to show from the above completeness relation that the components of the Riemann tensor can be expressed in terms of the real and imaginary parts of M_{ab} and on comparison of the result with (2.6) one arrives at (2.7).

Appendix 2

By substituting the condition $l^a R_{abcd} = l_b A_{cd}$ into the expression for the Weyl tensor:

$$C_{abcd} = R_{abcd} + R_{c[a}g_{b]d} + R_{d[b}g_{a]c} + \tfrac{1}{3}R g_{c[b}g_{a]d} \tag{A.1}$$

together with the type N condition $l^a C_{abcd} = 0$ and the fact that l^a is a Ricci eigenvector, one easily finds that in all cases, $R = 0$. In case (i), the conditions $l^a R_{abcd} = 0$, $l^a C_{abcd} = 0$ and $R = 0$ show that the Ricci tensor takes the form (2.1). Arguments similar to those concerning equation (3.2) show that a constant null bivector with principal null direction parallel to l is admitted. In case (ii) A_{ab} is a null bivector with principal null direction parallel to l. On contracting (A.1) with l^a and using $l^a C_{abcd} = 0$ and $R = 0$ one can check that the only eigendirections of the Ricci tensor are those contained in the

† For both conformally flat and type N null Maxwell fields one can use the Bianchi identities to show easily that the expansion and twist as well as the shear of the principal null congruence vanish. Only in the conformally flat case does the rotation also necessarily vanish.

(null) blade of $\overset{*}{A}_{ab}$ and that the respective eigenvalues are zero. Thus one arrives at Segré type {(3, 1)} with zero eigenvalue. In case (iii) A_{ab} is a non-null bivector with timelike blade. If one selects a null tetrad based on the principal null directions of A_{ab} then (A.1) when contracted with the members of this tetrad show that any (spacelike) direction in the blade of $\overset{*}{A}_{ab}$ is a Ricci eigendirection .The rest of the calculation is then fairly straightforward.

Appendix 3

The fields characterized by the equivalent conditions (a'), (b'), (c') and (d') in § 3 are a natural generalization of the p–p waves of Ehlers and Kundt. To see this, recall that Ehlers and Kundt (1962) gave the following characterizations of p–p wave space–times.

(1) A non-flat vacuum field is a p–p wave $\Leftrightarrow$ it admits a constant (necessarily null) bivector.

(2) A non-flat vacuum field is a p–p wave $\Leftrightarrow$ the Riemann tensor is complex recurrent (with recurrence 1-form necessarily a gradient).

(3) A non-flat vacuum field is a p–p wave $\Leftrightarrow$ it admits a constant (necessarily null) vector.

The Riemann tensor of a p–p wave is necessarily of Petrov type N.

The conditions (a'), (b') and (d') of the present paper correspond to the above conditions (1), (2) and (3) respectively. That the former conditions contain more restrictions is a consequence of the fact that *in vacuo* a complex recurrent space–time is necessarily of Petrov type N, necessarily has a gradient recurrence 1-form and the constant bivectors and vectors which are admitted are necessarily null. None of these results hold in general if the vacuum condition is dropped. General complex recurrent space–times could be either Petrov type N or D and have been separated into the four categories of type N (i), (ii) and (iii) and Type D. As was pointed out in § 3, the conditions of the type N (i) category are equivalent to the above conditions (a'), (b'), (c') and (d') and so this category selects what appears to be a fairly natural generalization of a p–p wave. The condition (c') shows how such a characterization may be given in terms of the algebraic properties of the Ricci tensor.

The equivalence of the conditions (a) and (b) given in § 3 is a more general extension of result (i) above.

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